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3 Spectral Distributions of Turbulence in a Plasma with Collisional and Collisionless Dissipations 6

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ABSTRACT

On the basis of the Navier-Stokes equation of motion, the equation of continuity for a compressible fluid, and the Poisson equation for charge densities, we investigate the spectral distributions of turbulent energy and density fluctuations in a plasma with a strong magnetic field. Collision dominated plasmas and collisionless plasmas are considered.

The nonlinear equations are solved by the use of a cascade decomposition, in which the big eddies contribute to the development of the energy spectra, and the small eddies set up turbulent properties of a medium in which the big eddies move. The turbulent motion of the big eddies is considered homogeneous, stationary, isotropic, and compressible. In the calculations of the turbulent properties, the motion of the small eddies is assumed incompressible, nonhomogeneous, anisotropic, nonstationary and a quasilinear approximation is applied to solve their dynamical equations.

For a modal transfer to prevail in the direction from big eddies to smaller ones, it is necessary to provide a dissipation, or a drain. Such a dissipation is represented by the viscosity in a collision dominated plasma. However, in a collisionless plasma, the above molecular dissipation does not exist, and its role is replaced by a collisionless dissipation, caused by the electrostatic correlations. It is found that the collisionless dissipation has also the form of a product of a vorticity by a diffusion coefficient, except the vorticity involves the density rather than the velocity, and the diffusion coefficient involves the Bohm diffusion, rather than the molecular viscosity. The formula of diffusion derived here confirms the Bohm formula

with a large numerical coefficient.

The study emphasizes a plasma where the electron temperature far exceeds the ion temperature, and the wave length of the turbulent motion exceeds the Debye length. However, the reduced system of fundamental equations, from which the spectra are derived are analogous to that governing gravity turbulence in atmosphere and ocean with a Coriolis force. This suggests that the method presented and some of the results obtained may be applicable to such problems too.

1. INTRODUCTION

We propose to derive the spectral distributions of turbulence and density fluctuations in a compressible plasma, with an externally applied constant magnetic field. The induced magnetic fluctuations, which are transversal electromagnetic waves are not taken into account. The plasma may be dominated by collisions or may be collisionless.

The fundamental equations of motion and charge density are reduced to a system of the type of the Riemann[1] equations, with the addition of a Coriolis force and a viscous force (Sec. 2). Some remarks on the solutions are given in Sec. 3.

In order to solve the nonlinear equations of turbulence, we introduce a method of "cascade decomposition" into big and small eddies, and apply a quasilinear approximation to the small eddies. As a result, the big eddies will determine the development of the spectra, and the small eddies will shape up the appropriate turbulent properties in the medium for the big eddies to evolve (Secs. 4,5).

After computations of the turbulent stresses and correlations from the solutions of the linearized equations of small eddies (Secs. 6,7), the spectral equations are derived (Sec. 8,9). A special emphasis is given to equilibrium

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[1] B. Riemann, Ueber die Fortpflantzung ebener Luftwellen von endlicher Schwingungsweite, Collected Works, 2nd edition, Edited by H. Weber, (Dover Publications, Inc. New York, 1953) .

turbulence (Sec. 9). The spectral equations appears to be governed by two types of functions: modal transfers and dissipations. They involve an eddy viscosity, an eddy diffusivity, and the Bohm diffusion, derived in Sec. 10. The eddy diffusivity agrees with the empirical formula of Heisenberg[2], but the eddy viscosity is different in view of the strong effect of the magnetic field. It is found that in a collisionless plasma, a dissipation can still be brought about by collective electrostatic fluctuations, in the form of a product of a vorticity with a diffusion coefficient, the value of which is found to agree with the Bohm diffusion. The spectral laws of turbulence and density F and G are derived for the following problems (Sec. 11):

(1) Convection of particles in a collisional plasma (Sec. 12.1)

$$F \sim k^{-2} \;;$$
$$G \sim k^{-3/2} \,.$$

(2) Diffusion of particles in a collisional plasma (Sec. 12.1)

$$F \sim k^{-2} \;;$$
$$G \sim k^{-9/2} \,.$$

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[2] W. Heisenberg, Z. f. Phys. 124, 628 (1948).

(3) Dissipation of turbulence and diffusion of particles in a collisional plasma (Sec. 12.2)

$$F \sim k^{-7} \quad ;$$

$$G \sim k^{-9} \quad .$$

(4) Diffusion of particles in a collisionless plasma (Sec. 13)

$$F \sim k^{-3} \quad ;$$

$$G \sim k^{-5} \quad .$$

Note the difference of results in the problem (2) with collision and the problem (4) without collision. The factors in front of the power laws, as well as the critical wave numbers characterizing the fall of the spectra, are also determined. Further discussions and applications to similar problems are found in Sec. (14).

2. REDUCED SYSTEM OF EQUATIONS FOR THE MOTION OF COLD IONS IN TWO DIMENSIONS

We consider ion waves in a plasma, where the electron temperature considerably exceeds the ion temperature. As the phase velocity of such waves is much greater than the ion thermal velocity, we can disregard the effects from the spread of the ion velocity distribution, and consider the moment equations for ions, in which the ion number density is

$$n_i = N + n \quad ,$$

where N is the mean number density, and n is the density fluctuation. In view of the high temperature and the high collision frequency, the electrons will have a maxwellian velocity distribution, and a density following the Boltzmann distribution, with n_i as the ambiant number density of a quasi-stationary nature. Thus the electron number density is

$$n_e = n_i \exp(\psi/a) \ ,$$

where ψ is an electrostatic potential defined by

$$\underset{\sim}{E} = - a \nabla\psi \ ,$$

and has the dimension of a velocity, $\underset{\sim}{E}$ is the electrostatic force, which is the electrostatic field multiplied by e/M , and

$$a = (K T_e/M)^{\frac{1}{2}}$$

is the phase velocity, based on the electron temperature T_e, and the ion mass M, with the Boltzmann gas constant K. Because of the small value of the exponent ψ/a, and the quasi-stationarity of the ambiant temperature n_i, we can write approximately

$$n_e \cong n_i(1 + \psi/a) .$$

The variation

$$\delta(n_i - n_e) = \frac{\partial(n_i - n_e)}{\partial n} \delta n + \frac{\partial(n_i - n_e)}{\partial \psi} \delta\psi$$

consists of a variation δn and a variation $\delta\psi$. The variation δn is contributed by the ions, with the electrons as forming a stochastic background, and the variation $\delta\psi$ is contributed by the electrons only. Thus

$$\delta(n_i - n_e) = \delta n - \frac{N+n}{a} \delta\psi ,$$

and the Poisson equation becomes

$$-\nabla \delta\psi = \frac{4\pi e^2}{Ma} \delta n - \frac{4\pi e^2 N}{Ma^2} \frac{N+n}{N} \delta\psi .$$

The case encountered in most experiments deals with wave lengths exceeding the Debye length. In this case the quasi-neutrality exists, the left hand side drops, and the right hand side yields

$$\delta\psi = a\delta\ell n(N+n). \tag{1}$$

In the following, we shall call ψ a density function, and its spectrum will be called density spectrum.

Now the equations of momentum and continuity for the ion motion are

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla\right)\underset{\sim}{u} = \underset{\sim}{E} + \frac{e}{M}\,\frac{\underset{\sim}{u} \times \underset{\sim}{B}_0}{c} + \nu\,\nabla^2\underset{\sim}{u} \quad ,$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla\right) n + (N+n)\,\nabla \cdot \underset{\sim}{u} = 0 \quad .$$

Upon introducing the function ψ from Eq. (1) , we can reduce the equations in the two-dimensional plane perpendicular to the external magnetic field as follows:

$$\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\right)u_i = -a\,\frac{\partial\psi}{\partial x_i} + \omega_c\, I_{ij}\, u_j + \nu\,\frac{\partial^2 u_i}{\partial x_j^2} \quad , \tag{2}$$

$$\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\right)\psi = -a\,\frac{\partial u_j}{\partial x_j} \quad ,$$

where the tensor I_{ij} has its components

$$I_{11} = I_{22} = 0 \quad ; \quad I_{12} = -I_{21} = 1 \; ,$$

and the cyclotron frequency is

$$\omega_c = eB_0/Mc \quad .$$

In the above momentum equation for the ion motion, we have neglected the ion pressure; this is justified because the ion temperature is negligible, compared to the electron temperature. An approximate viscous term,

$$\nu \nabla^2 \underset{\sim}{u} \; ,$$

has been added to represent a molecular damping of the ion motion, with a constant kinematic viscosity ν , which may be dropped in a collisionless plasma. Since the viscous effect is unimportant here, it is not the intention of writing the viscous term in full, including the compressibility.

The one-dimensional degeneration of the reduced system without the magnetic field is recognized as the Riemann equations, used in gravity waves and rarefaction waves. We observe that a magnetic field corresponds to a Coriolis force. Therefore the present treatment may be applicable to such problems too.

3. REMARKS ON THE THEORIES OF TURBULENCE

Most theories on the energy spectrum deal with an incompressible, isotropic, homogeneous and stationary turbulent fluid. For the purpose of simplifying the discussions, we may fix our attention to the following equation

$$\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) u_i = \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

or preferably its Fourier transform, used as a model of studying turbulence.

The inertia term, with the dimension u^2/ℓ (ℓ is a length, and $k = 1/\ell$ a wave number), plays an important role in the inertial portion of the universal spectrum, which is then governed by a constant transfer of energy between the modes or sizes (modal transfer) in an equilibrium turbulence. It is written dimensionally as

$$\frac{u^3}{\ell} = \text{constant}.$$

The spectrum F, related to the energy by

$$\tfrac{1}{2}\langle u^2 \rangle \equiv \int_0^\infty dk \;\; F \;,$$
$$= \text{const}\;\ell^{2/3} \;,$$

is found to be given by the formula

$$F = \text{const}\;k^{-5/3} \;,$$

which is the spectral law of Kolmogoroff[3].

A Fourier decomposition of the model equation gives a spectral equation, involving a viscous dissipation and a nonlinear modal transfer. If the latter is assumed to be equivalent to a new dissipation with an eddy diffusivity postulated dimensionally, the same spectral law is obtained. This is the method used by Heisenberg[2].

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[3] A.N. Kolmogoroff, Comp. Rend. Acad. Sci. URSS 30, 301(1941).

The mathematical problem of turbulence is, of course, much more complex. From the above model equation, or its Fourier transform, an equation of velocity correlation of any order can be generated, but it involves always a correlation of a higher order. Thus we have to deal with a hierarchy of correlation equations, the resolution of which requires a cutoff. A closed system is obtained, when the fourth order is approximated by a sum of products of second order correlations. Such an endeavor is undoubtedly tedious, but the results prove even more discouraging, when a negative spectrum is found by the method [4]. We shall not discuss other more complex methods.

The above failure can be attributed to the strong interactions between the four individual modes, so that they may not be decoupled. A decoupling would be permissible with an advanced state of randomness. We assume that the randomness is increased by the following procedures:

(1) Instead of individual modes, we consider groups of modes; within each group, and internal smoothing process secures a greater randomization.

(2) The velocity distribution in a turbulent motion deviates from its normal distribution by the presence of small eddies at high velocities. It follows that the group of big eddies in the cascade, being depleted of small eddies, becomes quasi-normal.

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[4] Y. Ogura, J. Fluid Mech. 16 , 33(1963).

(3) The weak interaction between two separate groups justifies a decoupling.

In the following we shall consider a simple cascade consisting of two groups. It will be shown that the group of small eddies contribute in shaping eddy transport properties of the medium in which the big eddies are to move, and the big eddies are responsible for the evolution of the energy spectrum. The small eddies, containing a negligible amount of energy, will be treated by a quasilinear approximation, and the big eddies, as determined by the hierarchy, will be sufficiently random to justify the decoupling of correlations.

In the framework of hydrodynamic turbulence, the method conveniently bypasses the hierarchy and derives in a straight forward way the spectral equation of Heisenberg[2] and the eddy viscosity.

In the present problem of plasma turbulence, the hierarchy of big eddies will retain mixed correlations of big and small eddies to the fourth order, and correlations of big eddies to the second order. If the higher order correlations of big eddies are assumed to have a negligible contribution to the universal range of spectrum (small eddies), the hierarchy becomes closed.

4. METHOD OF CASCADE DECOMPOSITION

We write the velocity into two parts:

$$u(\underset{\sim}{x}) = u_0(\underset{\sim}{x}) + u'(\underset{\sim}{x}) \ , \tag{3a}$$

with

$$u_0(x) = \int_0^k d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\,\underset{\sim}{x}}\, u(\underset{\sim}{x}) \quad , \tag{3b}$$

and

$$u'(x) = \int_k^\infty d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\,\underset{\sim}{x}}\, u(\underset{\sim}{x}) \quad , \tag{3c}$$

representing the big and small eddies respectively. Here

$$\int_{\underset{\sim}{k}=-\infty}^{\infty} d\underset{\sim}{k} = \int_{k=0}^{\infty} dk \quad ,$$

$$= \int_0^k dk + \int_k^\infty dk \quad ,$$

the last two integrals being volume integrals respectively within and outside a sphere of radius k . The same notations with indicies ($_0$) and (') will be applied to the variable ψ . In this way the equations (2) determining $\underset{\sim}{u}$ and ψ are decomposed into the following :

For the big eddies ,

$$\frac{du_{0i}}{dt} = -\,a\,\frac{\partial\psi_0}{\partial x_i} + \omega_c\, I_{ij}\, u_{0j} + \nu\,\frac{\partial^2 u_{0i}}{\partial x_j^2} - < u_j' \frac{\partial u_i'}{\partial x_j} >_k \quad ,$$

$$\frac{d\psi_0}{dt} = -\,a\,\frac{\partial u_{0j}}{\partial x_j} - < u_j' \frac{\partial \psi'}{\partial x_j} >_k \quad , \qquad (4)$$

$$E_{0i} = -\,a\,\frac{\partial \psi_0}{\partial x_i} \quad ,$$

and for the small eddies ,

$$\frac{du_i'}{dt} = -\,a\,\frac{\partial\psi'}{\partial x_i} + \omega_i I_{ij}\, u_j' - u_j' \frac{\partial u_{0i}}{\partial x_j}$$

$$+ \left[\nu\,\frac{\partial^2 u_i'}{\partial x_j} - u_j' \frac{\partial u_i'}{\partial x_j} + < u_j' \frac{\partial u_i'}{\partial x_j} >_k \right] \quad ,$$

$$\frac{d\psi'}{dt} = -\,u_j' \frac{\partial \psi_0}{\partial x_j} + \left[-a\,\frac{\partial u_j'}{\partial x_j} - u_j' \frac{\partial \psi'}{\partial x_j} + < u_j' \frac{\partial \psi'}{\partial x_j} >_k \right] \quad , \qquad (5)$$

$$E_i' = -\,a\,\frac{\partial \psi'}{\partial x_i} \quad ,$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_{0j}\,\frac{\partial}{\partial x_j} \quad .$$

The terms

$$< u_j' \frac{\partial u_i'}{\partial x_j} >_k \qquad \text{and} \qquad < u_j' \frac{\partial \psi'}{\partial x_j} >_k \tag{6}$$

in Eqs. (4) are called eddy stresses, representing the effects of the motion of small eddies upon the bigger ones, and they are to be calculated from Eqs. (5).

The motion of the small eddies are essentially very different from the bigger ones. Therefore separate assumptions are introduced. In studying the motion of the small eddies, the assumptions are :

(i) Small eddies move rapidly in a quasi-stationary background

Choose a length scale k^{-1} separating the two groups of eddies. An average over such a length interval, denoted by

$$< \ldots >_k$$

will average out the fluctuations of the small eddies, but leaves intact the motion of the big eddies. However, an average over a long interval of length $\ell_0 \to \infty$, denoted by

$$< \ldots >$$

will even average out the fluctuations of big eddies.

The application of the averages enable us to derive the separate equations of motions for the small and big eddies, from the equation (2) of the total motion.

With respect to the rapidly varying motion of the small eddies, their background motion, as provided by the bigger ones, can be considered as quasi-stationary, i.e. it varies slowly in time and space. Thus the motion of the small eddies is anisotropic, inhomogeneous, and non-stationary.

(ii) <u>Quasilinearization of the equations of motion of the small eddies</u>

The small eddies contribute in eddy transport properties, and do not embody the major energy. Therefore they are studied by a quasilinear approximation. It should be valid in the universal spectrum, i.e. for sufficiently large k .

(iii) <u>Small eddies are incompressible and inviscid</u>

The main compressibility effect is exhibited by the big eddies , although the small eddies have a secondary compressibility effect in their role of producing eddy stresses, this effect together with the damping are neglected.

Now the assumptions concerning the motion of the big eddies are:

(iv) <u>The turbulent motion of the big eddies is isotropic, stationary and homogeneous</u>

(v) <u>The big eddies move in a locally isotropic and homogeneous medium</u>

The small eddies prescribe eddy transport properties, which are locally isotropic and homogeneous.

5. SIMPLIFIED EQUATIONS OF MOTION

With the use of the assumption of quasilinearization (ii), the assumption of incompressible and inviscid small eddies (iii), the equations of motion of small eddies (5) are simplified, by dropping the terms between the brackets, giving

$$\frac{du_i'}{dt} - \omega_c\, I_{ij}\, u_j' = -a \frac{\partial\psi'}{\partial x_i} - u_j' \frac{\partial u_{0i}}{\partial x_j} \quad ,$$

$$\frac{d\psi'}{dt} = -u_j' \frac{\partial\psi_0}{\partial x_j} \quad . \tag{7}$$

The equations of motion of the big eddies (4) remain nonlinear, compressible and viscous.

By multiplying the two equations (4) by u_{0i} and ψ_0 respectively, we formulate the following energy equations

$$\tfrac{1}{2} \frac{\partial}{\partial t} < u_0^2 > = < \underset{\sim}{E}_0 \cdot \underset{\sim}{u}_0 > - \nu < \left(\frac{\partial u_{0i}}{\partial x_j} \right)^2 > - < u_{0i} < u_j' \frac{\partial u_i'}{\partial x_j} >_k >$$

$$\tfrac{1}{2} \frac{\partial}{\partial t} < \psi_0^2 > = -a < \frac{\partial u_{0i}}{\partial x_j} \psi_0 > - < \psi_0 < u_j' \frac{\partial \psi'}{\partial x_j} >_k > + \frac{1}{a} < \underset{\sim}{u}_0 \cdot \underset{\sim}{E}_0 \psi_0 > . \tag{8a}$$

On the right hand side of the last equation, the first term is transformed into $- < \underset{\sim}{E}_0 \cdot \underset{\sim}{u}_0 >$, and the last term, which contributes most to the non-universal range (big eddies) of the spectrum, is assumed negligible in the universal range. Hence

$$\frac{1}{2} \frac{\partial}{\partial t} < \psi_0^2 > \; = \; - < \underset{\sim}{E}_0 \cdot \underset{\sim}{u}_0 > - < \psi_0 < u_j' \frac{\partial \psi'}{\partial x_j} >_k > \tag{8b}$$

As shown later in Sec. 7, the diffusion

$$- < \underset{\sim}{E}_0 \cdot \underset{\sim}{u}_0 >$$

can be calculated on the basis of

$$< \underset{\sim}{E}' \cdot \underset{\sim}{u}' > \; ,$$

without the intermediary of hierarchy of equations of big eddies.

The eddy stresses involved are calculated by means of the equations of motion of the small eddies (7), giving the solutions:

$$\begin{aligned} u_i'(t) &= - \frac{\partial u_{0s}}{\partial x_j} \int_{t_0}^{t} dt' \, P_{si}(t-t') \, u_j'(t') + \int_{t_0}^{t} dt' \, P_{si}(t-t') \, E_s'(t'), \\ \psi'(t) &= - \frac{\partial \psi_0}{\partial x_j} \int_{t_0}^{t} dt' \, u_j'(t') \, . \end{aligned} \tag{9}$$

In the above Lagrangian formulation, we have written

$$u'_j(t') \quad \text{to represent} \quad u'_j[t', \underset{\sim}{x}'(t')] .$$

The same representation holds for $E'_s(t')$. Further,

$$P_{si}(t) = \delta_{si} \cos \omega_c t + I_{si} \sin \omega_c t$$

and $t_o \to \infty$. The initial value at $t_0 \to \infty$ are dropped, as they are not correlated with their later values at t. It is to be noticed that the term P_{si} for $s \neq i$ will not contribute in the following analysis.

6. EDDY STRESSES

The eddy stresses (6), as occuring in Eqs.(4) and (8) are now computed from the solutions (9). Evidently we shall expect to obtain a large scale contribution (which does not vanish by a large scale average) and a small scale contribution (which vanishes by a large scale average). We shall be concerned with the latter contribution here, while the former will be referred to Sec. 7. The stresses play an important role in the modal transfers across the spectra.

6.1 Calculation of the Turbulent Stress $< u_i' \, u_j' >_k$

In the evaluation of the turbulent stress

$$< u_i' \, u_j' >_k = - \frac{\partial u_{0s}}{\partial x_r} \int_{t_0}^{t} dt' \, P_{si}(t-t') < u_r'(t') u_j'(t) >_k$$

$$+ \int_{t_0}^{t} dt' \, P_{si}(t-t') < E_s'(t') \, u_j'(t) >_k \quad , \tag{10}$$

from (9), we may infer that the two integrals are associated with the gradients

$$\frac{\partial u_{0s}}{\partial x_r} \quad \text{and} \quad \frac{\partial E_{0s}}{\partial x_r}$$

respectively.

In view of the local stationarity and homogeneity, the first integral is, according to (v) :

$$\int_{t_0}^{t} dt' \, P_{si}(t-t') < u_r'(t') \, u_j'(t) >_k$$

$$= \nu_k \, \delta_{si} \, \delta_{rj} \quad ; . \tag{11}$$

where

$$\nu_k = \tfrac{1}{2} \int_0^\infty d\tau \quad \cos \quad \omega_c \tau < \underset{\sim}{u}'(0) \cdot \underset{\sim}{u}'(\tau) >_k \quad . \tag{12}$$

By using (9), the second integral of (10) is explicitly written as

$$\int_{t_0}^{t} dt' \; P_{si}(t-t') < E_s'(t') \; u_j'(t) >_k$$

$$= - \frac{\partial E_{0s}}{\partial x_r} \int_{t_0}^{t} dt' \; P_{si}(t-t') \int_{t_0}^{t'} dt'' < u_r'(t'') \; u_j'(t) >_k \tag{13}$$

where the double integral is transformed as follows:

$$\int_{t_0}^{t} dt' \quad \int_{t_0}^{t'} dt'' \; P_{si}(t-t') < u_r'(t'') u_j'(t) >_k$$

$$= \quad \int_0^\infty d\tau' \quad \int_{\tau'}^\infty d\tau'' \; P_{si}(\tau') < u_r'(0) \; u_j'(\tau'') >_k \quad ,$$

$$t-t' = \tau' \quad , \qquad t-t'' = \tau'' \quad ; \tag{14}$$

$$= \quad \int_0^\infty d\tau'' < u_r'(0) \; u_j'(\tau'') >_k \int_0^{\tau''} d\tau' \; P_{si}(\tau') \quad ,$$

$$= \omega_c^{-1} \int_0^\infty d\tau'' < u_r'(0)\; u_j'(\tau'') >_k \quad (1 - \cos \omega_c\, \tau'')\, \Delta_{si} \quad ,$$

$$= (\lambda_k - \nu_k)\; \omega_c^{-1}\, \Delta_{si}\, \delta_{rj} \quad ; \qquad (15)$$

with

$$\Delta_{si} = \begin{cases} 1 \;, & s \neq i \;, \\ 0 \;, & s = i \;; \end{cases}$$

$$\delta_{rj} = \begin{cases} 1 \;, & r = j \;, \\ 0 \;, & r \neq j \;; \end{cases}$$

and

$$\lambda_k = \tfrac{1}{2} \int_0^\infty d\tau < \underset{\sim}{u'}(0) \cdot \underset{\sim}{u'}(\tau) > . \qquad (16)$$

Hence after substitution of (11) - (16) into (10), we find

$$- < u_i'\; u_j' > \;=\; \nu_k \frac{\partial u_{0s}}{\partial x_j}\, \delta_{si} + (\lambda_k - \nu_k) \frac{\partial E_{0s}}{\partial x_j}\; \omega_c^{-1}\, \Delta_{si} \quad ;$$

and consequently

$$< u_{0i} \frac{\partial}{\partial x_j} < u_i'\; u_j' >_k > \;=\; - << u_i'\; u_j' >_k \frac{\partial u_{0i}}{\partial x_j} > \quad ,$$

$$= \nu_k\, R_0 + (\lambda_k - \nu_k)\, \Gamma_0 \quad ;$$

where the notations R_0 , Γ_0 , representing vorticities, are introduced for the sake of abbreviation of writings. They are

$$R_0 = < (\frac{\partial u_{0i}}{\partial x_j})^2 > ,$$

$$\Gamma_0 = < \frac{\partial E_{0s}}{\partial x_j} \frac{\partial u_{0i}}{\partial x_j} > \omega_c^{-1} \Delta_{si} ,$$

$$= < \frac{\partial E_{0i}}{\partial x_j} \frac{\partial u_{0s}}{\partial x_j} > \omega_c^{-1} \Delta_{si} .$$

We remark that Γ_0 results from a shear correlation, and can be dropped in the assumption of the isotropy of the big eddies. Hence we find simply

$$< u_{0i} \frac{\partial}{\partial x_j} < u_i' u_j' >_k > = \nu_k R_o , \tag{17}$$

which is the modal transfer function in the development of the turbulent spectrum.

6.2 <u>Calculation of the Density Stress</u> $< u_j' \partial\psi'/\partial x_j >_k$

With the aid of (9), and the assumption (v), we find simply

$$< u_j' \frac{\partial \psi'}{\partial x_j} >_k = \frac{\partial}{\partial x_j} < u_j' \psi' >_k ,$$

$$= - \lambda_k \frac{\partial^2 \psi_0}{\partial x_j^2} ;$$

and consequently

$$< \psi_0 < u_j' \frac{\partial \psi'}{\partial x_j} >_k > = \lambda_k J_0 , \tag{18}$$

which is the modal transfer function in the development of the density spectrum. Here

$$J_0 = < (\frac{\partial\psi_0}{\partial x_j})^2 >$$

is the vorticity function for ψ_0 .

7. ELECTROSTATIC DIFFUSION

The time evolution of the turbulent energy and the density fluctuations (8) is governed by the eddy dissipations (17) and (18), and the electrostatic diffusion

$$< \underset{\sim}{E}_0 \cdot \underset{\sim}{u}_0 > \quad .$$

We note that the second of Eqs. (4) for the density fluctuations can be rewritten in the form

$$a \frac{\partial\psi_0}{\partial t} - \underset{\sim}{E}_0 \cdot \underset{\sim}{u}_0 = - a^2 \frac{\partial u_{0j}}{\partial x_j} + < \underset{\sim}{E}' \cdot \underset{\sim}{u}' >_k$$

By taking a large scale average, we find

$$\begin{aligned} - < \underset{\sim}{E}_0 \cdot \underset{\sim}{u}_0 > &= << \underset{\sim}{E}' \cdot \underset{\sim}{u}' >_k > \quad , \\ &= < \underset{\sim}{E}' \cdot \underset{\sim}{u}' > , \\ &= \Phi_k \, . \end{aligned}$$

Thus the calculation of the diffusion is reduced to the calculation of the correlation $< \underset{\sim}{E}' \cdot \underset{\sim}{u}'>$ of the small eddies, bypassing the necessity of going into the hierachy of equations. The calculations are based upon (9) and will thus select the non-vanishing contributions from the large scale averages. We shall rewrite the second of Eqs. (9) in the form

$$E_i'(t) = - \frac{\partial E_{0i}}{\partial x_j} \int_{t_0}^{t} dt' \; u_j'(t') \; .$$

Taking the product between the two equations (9), we obtain the large scale average

$$\begin{aligned} \Phi_k &= < \underset{\sim}{E}' \cdot \underset{\sim}{u}' > \\ &= < \frac{\partial E_{0i}}{\partial x_j} \frac{\partial u_{0s}}{\partial x_m} > \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \; P_{si}(t-t') < u_m'(t') \; u_j'(t'') > \\ &\quad + \int_{t_0}^{t} dt' \; P_{si}(t-t') < E_s'(t') \; E_i'(t) > \quad . \end{aligned}$$

After a change of variables (14), the double integral becomes

$$\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \; P_{si}(t-t') < u_m'(t') \; u_j'(t'') >_k \quad .$$

$$= \int_{0}^{\infty} d\tau' \int_{0}^{\infty} d\tau'' \; P_{si}(\tau') < u_m'(0) \; u_j(\tau' - \tau'') >_k \quad ,$$

$$= \int_{-\infty}^{\infty} d\eta \int_{\eta}^{\infty} d\tau' \, P_{si}(\tau') < u_m'(0) \, u_j'(\eta) >_k \quad ,$$

$$= 2\nu_k \, \omega_c^{-1} \, \Delta_{si} \, \delta_{mj} \quad .$$

Hence, after substitution, we obtain

$$\Phi_k = \int_0^{\infty} d\tau < \underset{\sim}{E}'(0) \cdot \underset{\sim}{E}'(\tau) >_k \cos \omega_c \tau + 2\nu_k \, \Gamma_0 \quad .$$

If Γ_0 , which depends on the shear correlation between $\underset{\sim}{E}_0$ and $\underset{\sim}{u}_0$, is again left out, on account of the isotropy, Φ_k simplifies to

$$\Phi_k = \int_0^{\infty} d\tau < E'(0) \, E'(\tau) >_k \cos \omega_c \tau \quad , \tag{19}$$

and becomes the electrostatic diffusion in the velocity space.

8. SPECTRAL FUNCTIONS

Upon substituting from Eqs. (17) - (19) of the stresses and the diffusion into Eqs. (8), we obtain the following equations for the energy and the density fluctuations:

$$-\frac{\partial}{\partial t} < u_0^2 > \; = \; (\nu + \nu_k)\, R_0 + \Phi_k \; , \qquad (20)$$

$$-\frac{\partial}{\partial t} < \psi_0^2 > \; = \; \lambda_k \, J_0 - \Phi_k \; ;$$

where ν_k , λ_k and Φ_k have been defined earlier by Eqs. (12), (16) and (19) .

In terms of the Fourier representation (3b), the energy is

$$\tfrac{1}{2} < u_0^2 > \; = \; \tfrac{1}{2} \lim_{\ell_0 \to \infty} \int_0^k d\underset{\sim}{k} \left(\frac{\pi}{\ell_0} \right)^2 \underset{\sim}{u}(\underset{\sim}{k}) \cdot \underset{\sim}{u}(\underset{\sim}{k}) \; ,$$

$$= \int_0^k dk \, F(k) \; ;$$

where F is the spectral function of turbulence. In the same representation we find the vorticity

$$R_0 \; = \; 2 \int_0^k dk \, k^2 \, F \; .$$

The density counterparts are

$$\tfrac{1}{2} < \psi^2 > \; = \int_0^k dk \, G(k) \; ,$$

$$J_0 \; = \; 2 \int_0^k dk \, k^2 G \; .$$

The electrostatic energy in terms of the potential function is:

$$< E_0^2 > \; = \; a^2 < \left(\frac{\partial \psi_0}{\partial x_j} \right)^2 > \quad ,$$

$$= \; a^2 \, J_0 \quad .$$

For $k = \infty$, we reduce to

$$R_0(k = \infty) \; = \; R \quad ,$$

and

$$J_0(k = \infty) \; = \; J \quad .$$

9. SPECTRAL EQUATIONS FOR EQUILIBRIUM TURBULENCE

We shall express the equations of turbulent energy and density fluctuations (20) in terms of spectral functions. That portion of the spectrum, related to the wave numbers smaller than the universal range, depends on the size of the experimental vessel, and will not be considered here.

Further we introduce the rate of viscous dissipation

$$\varepsilon_\nu = \nu R \quad .$$

Then we can reduce Eqs. (20) to the following

$$(\nu + \nu_k)\, R_0 + \Phi_k = \varepsilon_\nu \quad , \tag{21a}$$

$$\lambda_k\, J_0 - \Phi_k = 0 \quad . \tag{21b}$$

The terms

$$\nu_k\, R_0 \quad \text{and} \quad \lambda_k J_0$$

represent the modal transfers, and the term Φ_k represents the electrostatic diffusion, which produces electrostatic fluctuations by compressibility at the expense of the turbulent kinetic energy. Thus we see that Φ_k , which occurs in opposite signs in Eqs. (21a) and (21b) plays the role of a new turbulent dissipation in the turbulent energy equation (21a), and the role of a supply of energy to set up density fluctuations in the density equation (21b).

10. EDDY VISCOSITY, EDDY DIFFUSIVITY AND BOHM DIFFUSION

10.1. Eddy Viscosity and Diffusivity

Applying a Fourier transform, we can write

$$< [u(t,\underset{\sim}{x})]^2 > = \left(\frac{\pi}{\ell_0}\right)^2 \int_{-\infty}^{\infty} d\underset{\sim}{k}\, |u(t,\underset{\sim}{k})|^2 \quad ,$$

$$= 2 \int_0^{\infty} dk\, F(t,\underset{\sim}{k}) \quad ;$$

where

$$F(t,\underset{\sim}{k}) = \tfrac{1}{2}\left(\frac{\pi}{\ell}\right)^2 2\pi k\,|u(t,\underset{\sim}{k})|^2$$

is the spectral function of turbulence, and $2\ell_0$ is the interval of length used in the average.

In a similar way, we have

$$< \underset{\sim}{u}'(t',\underset{\sim}{x}')\underset{\sim}{u}'(t,\underset{\sim}{x}) > \;=\; \frac{\pi}{T}\left(\frac{\pi}{\ell}\right)^2 \int_{-\infty}^{\infty} d\underset{\sim}{k} \int_{-\infty}^{\infty} d\omega\,|\underset{\sim}{u}'(\omega,\underset{\sim}{k})|^2\, e^{-i(\omega-\underset{\sim}{k}\,\underset{\sim}{u}_0)\tau} \;.$$

Here

$$\tau = t - t',$$

$$\underset{\sim}{x} = \underset{\sim}{x}' + \underset{\sim}{k}\underset{\sim}{u}_0 \;.$$

Thus

$$\nu_k = \frac{1}{4}\int_{-\infty}^{\infty} d\tau < \underset{\sim}{u}'(t',\underset{\sim}{x}')\;\underset{\sim}{u}'(t,\underset{\sim}{x}) > \cos\omega_c\tau \;,$$

$$= \frac{1}{4}\;\frac{\pi}{T}\left(\frac{\pi}{\ell}\right)^2 \int_{-\infty}^{\infty} d\underset{\sim}{k}\int_{-\infty}^{\infty} d\omega\,|\underset{\sim}{u}'(\omega,\underset{\sim}{k})|^2 \int_{-\infty}^{\infty} d\tau\; e^{-i(\omega-\underset{\sim}{k}\,\underset{\sim}{u}_0)\tau}\cos\omega_c\tau$$

$$= \frac{\pi}{2}\;\frac{\pi}{T}\left(\frac{\pi}{\ell}\right)^2 \int_{k}^{\infty} dk\; 2\pi k\,|\underset{\sim}{u}(\omega_k,\underset{\sim}{k}|^2 \;; \qquad (22)$$

with

$$\omega_k = |\underset{\sim}{k}\,\underset{\sim}{u}_0 \pm \omega_c| \;.$$

In the inertial subrange, the spectrum

$$\frac{\pi}{T} \left| \underset{\sim}{u} (\omega_k , \underset{\sim}{k}) \right|^2$$

may be assumed to follow a power law

$$\omega_k^{-m} .$$

If furthermore , we assume, as done usually, that the inertial subrange represents the energy containing portions of the spectrum, then

$$\frac{\pi}{T} \left| u(\omega_k, k) \right|^2 = \chi(m-1) < |u(t,k)|^2 > \omega_k^{-1} ,$$

where χ is a numerical coefficient of the order of unity. By the application of the Kolmogoroff[2] similitude consideration, the numerical coefficient m can be also determined and is

$$m = 2 .$$

Thus upon substitution into Eq. (22) , we find

$$\nu_k = \chi_1 \int_k^{\infty} dk \frac{F(t,k)}{\omega_k} ; \qquad (23)$$

with

$$\chi_1 = \pi \chi .$$

Now ω_k will be evaluated approximately. Writing it in full in Eq. (23), we have

$$\nu_k = \chi_1 \int_k^\infty dk' \, F(t,k') \, [\omega_c^2 + 2k'^2 \int_0^{k'} dk'' \, F(k'')]^{-\frac{1}{2}} .$$

The integrand consists of a factor $F(t,k')$ decreasing rapidly with k', and a factor with brackets which varies slowly in k'. Assuming a quasi-stationary spectrum

$$F(k'') \cong F(k') \quad ,$$

within the brackets, we obtain

$$\nu_k = \frac{\chi_1}{\omega_c} \int_k^\infty dk' \, F(t,k') \quad , \quad \omega_c >> R_0 , \qquad (24a)$$

valid in the inertial subrange, and

$$\nu_k = \chi_2 \int_k^\infty dk' \, [F(k')/k'^3]^{\frac{1}{2}} , \quad \omega_c << R_0 , \qquad (24b)$$

$$\chi_2 = \chi_1/\sqrt{2} \quad ,$$

valid in the dissipative subrange.

The eddy diffusivity is obtained by putting $\omega_c = 0$, and is

$$\lambda_k = \chi_2 \int_k^\infty dk' \, [F(k')/k'^3]^{\frac{1}{2}} . \qquad (25)$$

The formula (25) is in agreement with that proposed empirically by Heisenberg[2].

10.2 Bohm Diffusion

The above arguments and calculations used for ν_k and λ_k can be repeated for the evaluation of the electrostatic diffusion Φ_k in the velocity space. We find

$$\Phi_k = 2\chi_1 a^2 \int_k^\infty dk' \, k'^2 \, G(k') \, \omega_{k'}^{-1} ,$$

$$= \begin{cases} \lambda(J-J_0) , & \text{when} \quad \omega_c \gg R_0 . \qquad (26a) \\ \sqrt{2}a^2 \int_k^\infty dk \, k^{\frac{1}{2}} \, GF^{-\frac{1}{2}} , & \text{when} \quad \omega_c \ll R_0 , \qquad (26b) \end{cases}$$

The coefficient of diffusion λ is found to be

$$\lambda = \chi_1 a^2 \omega_c^{-1} . \qquad (27)$$

It does not arise from the molecular motion, but from electrostatic and collective fluctuations. It agrees with the formula found by Bohm[5]. However, the numerical coefficient here is greater than the Bohm coefficient 1/16.

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[5] D. Bohm, "The Characteristics of Electrical Discharges in Magnetic Fields", edited by A. Guthrie and R.K. Wakerling, Chap. 2, Sec. 5. (McGraw-Hill Book Company, Inc., New York, 1949).

The existence of the coefficient of Bohm diffusion λ entails a rate of dissipation

$$\varepsilon_\lambda = \lambda J$$

for the density spectrum, similarly to the rate of dissipation ε_ν for the turbulent spectrum.

11. CASES OF SPECTRA WITH AND WITHOUT COLLISIONS

We shall distinguish the following subranges in a collisional and a collisionless plasma. The collision is represented by the molecular viscosity ν . It is to be noted that in a collisionless plasma, the density spectrum may be in the diffusive subrange, with a Bohm diffusion originated from electrostatic fluctuations.

(a) Inertial and Convective Subranges in a Collisional Plasma

Here, as the collision is dominant

$$\varepsilon_\nu \gg \varepsilon_\lambda \quad , \quad \text{or} \quad R \gg J \; .$$

The wave numbers are small such that

$$\nu_k \gg \nu \quad , \quad \text{and} \quad \lambda_k \gg \lambda \; ,$$

and the spectra F and G are not influenced by ν and λ .

(b) Inertial and Diffusive Subranges in a Collisional Plasma

Here again

$$\mathcal{E}_{\nu} \gg \mathcal{E}_{\lambda} \quad , \quad \text{or} \quad R \gg J \quad .$$

Now we consider the development of the density spectrum in a diffusive subrange range,

$$\lambda_k << \lambda \quad ,$$

under the background of a turbulent spectrum in its inertial subrange

$$\nu_k >> \nu \quad .$$

(c) Dissipative and Diffusive Subranges in a Collision Dominated Plasma

As the collision prevails, we have

$$\mathcal{E}_{\nu} >> \mathcal{E}_{\lambda} \quad , \quad \text{or} \quad R >> J \quad ,$$

too. Now both spectra are in the subranges of high wave numbers:

$$\nu_k << \nu \ , \quad \text{and} \quad \lambda_k << \lambda \quad .$$

(d) Diffusion of Particles in a Collisionless Plasma

Here as the collision is small or absent, we have

$$\varepsilon_\lambda >> \varepsilon_\nu \quad .$$

Like in case (b), we consider here also the development of the density spectrum in a diffusive subrange

$$\lambda_k << \lambda \quad ,$$

under the background of a turbulent spectrum in its inertial subrange

$$\nu_k >> \nu \quad .$$

In all the inertial cases (a), (b) and (d), the turbulent spectrum is in the inertial subrange, so that

$$R_0 << \omega_c^2 \quad ,$$

and ν_k is determined by the formula (24a), so that the spectral equations (21) simplify to

$$\begin{aligned} \nu_k R_0 + \lambda_k J_0 &= \varepsilon_\nu \quad , \\ (\lambda + \lambda_k) J_0 &= \varepsilon_\lambda \quad . \end{aligned} \qquad (28)$$

(case d)

The terms

$$\nu_k R_0 \quad \text{and} \quad \lambda_k J_0$$

are called modal transfer for turbulent and density spectra respectively. The terms

$$\nu R_0 \quad \text{and} \quad \lambda J_0$$

are the collisional dissipation by viscosity and the collisionless dissipation by Bohm diffusion respectively. Finally

$$\mathcal{E}_\nu \quad \text{and} \quad \mathcal{E}_\lambda$$

are the rate of dissipations for the turbulent and density spectra.

In the collisional case (a) and (b), further simplifications can be made by noting that

$$\lambda_k J_0 = \lambda(J - J_0) << \mathcal{E}_\nu \; .$$

When this term is neglected, the system (28) reduces to

$$\begin{aligned} \nu_k R_0 &= \mathcal{E}_\nu \; , \\ (\lambda + \lambda_k) J_0 &= \mathcal{E}_\lambda \; . \end{aligned} \tag{29}$$

(cases a, b)

The turbulent spectrum is completely determined by its own modal transfer, to be drained directly into a molecular dissipation, without the *intermediary of an electrostatic diffusion* .

In the case (c), as the turbulent spectrum is in the dissipative sub-range, we have

$$R_0 >> \omega_c^2$$

and ν_k , λ_k are governed by the formulas (24b) and (25), so that the spectral equations (21) becomes

$$(\nu + \lambda_k) R_0 + \lambda_k J_0 = \varepsilon_\nu ,$$

$$\lambda_k J_0 = \sqrt{2} a^2 \int_k^\infty dk \quad k^{\frac{1}{2}} GF^{-\frac{1}{2}} \qquad (30)$$

(case c) .

12. COLLISIONAL PLASMAS

12.1 Convection and Diffusion of Particles in a Turbulent Plasma with an Inertial Spectrum (Cases a and b)

The convective and diffusive subranges of the density spectra under an inertial turbulent spectrum, coverning cases (a) and (b) are governed by Eqs. (29). They can be solved separately. We find

$$F = \alpha k^{-2} . \qquad (31a)$$

This spectral law will fall by the molecular dissipation at the critical wave number

$$k_\nu = (\varepsilon_\nu / \omega_c \nu^2)^{\frac{1}{2}} \quad .$$

Further we find

$$G = \frac{9}{8\chi_2} \varepsilon_\lambda \alpha^{-\frac{1}{2}} \left[1 + (\frac{k}{k_B})^{3/2}\right]^{-2} k^{-3/2} ,$$

$$\simeq \begin{cases} \frac{9}{8\chi_2} \varepsilon_\lambda \alpha^{-\frac{1}{2}} k^{-3/2} , & \text{for } k << k_B ; \\ \frac{\chi_2}{2} \frac{\varepsilon_\lambda \alpha^{\frac{1}{2}}}{\lambda^2} k^{-9/2} , & \text{for } k >> k_B , \end{cases} \tag{31b}$$

where

$$\alpha = (\frac{\varepsilon_\nu \omega_c}{2\chi_1})^{\frac{1}{2}} \quad ;$$

and

$$k_B = (\frac{4\alpha\chi_2^2}{9\lambda^2})^{1/3}$$

is a critical wave number, characterizing the transition from the inertial subrange to the diffusive subrange.

12.2 Dissipative and Diffusive Subranges (case c)

By differentiating the system (30) with respect to k , the derivative being indicated by ('), we have the equations

$$\lambda_k' R_0 + \lambda_k' J_0 + (\nu + \nu_k) R_0' + \lambda_k J_0' = 0 ,$$

$$\lambda_k' J_0 + \lambda_k J_0' = \sqrt{2} a^2 k^{\frac{1}{2}} G F^{-\frac{1}{2}} ,$$

which, upon replacing

$$R_0, J_0 \text{ by } R, J ,$$

and neglecting λ_k, J as compared to ν , R , are reduced to

$$\lambda_k' R + \nu R_0' = 0 ,$$

$$\lambda_k' J = - \sqrt{2} a^2 k^{\frac{1}{2}} G F^{-\frac{1}{2}} .$$

The asymptotic solutions for large k are

$$F = (\chi_2 R / 2\nu)^2 k^{-7} , \qquad (32)$$

$$G = \chi_2 (\chi_2 R / 2\nu)^2 \frac{J}{\sqrt{2a^2}} k^{-9} .$$

13. DIFFUSION OF PARTICLES IN A COLLISIONLESS PLASMA (Case d)

Like in case (b), treated in Sec. 12, the problem is governed by the same equations (28). However, the modal transfer is now not drained by a molecular dissipation, absent in the present case, but by a Bohm diffusion.

We write the system (28) in the following form

$$\lambda J_0 - \nu_k R_0 = \lambda J \ ,$$
$$(\lambda + \lambda_k) J_0 = \lambda J \ . \tag{33}$$

In order to resolve the system (33), we differentiate with respect to k , and obtain

$$\lambda J_0' - \nu_k R_0' - \nu_k' R_0 = 0 \ ,$$

$$\lambda J_0' + \lambda_k J_0' + \lambda_k' J_0 = 0 \ .$$

For the diffusion to be effective, the processes in J_0 must have developed to sufficiently large wave numbers, so that we can replace J_0 by J, and neglect

$$\lambda_k << \lambda \ ,$$

and R_0 in the inviscid F spectrum. Thus we reduce to

$$\lambda J_0' - \nu_k R_0' = 0 \ ,$$

$$\lambda J_0' + \lambda_k' J = 0 \ .$$

The solutions are

$$F = \left(\frac{X_2}{X_1} J\omega_c \right)^{2/3} k^{-3} \quad ,$$

$$G = \frac{1}{2a^2} \left(\frac{X_2}{X_1} J\omega_c \right)^{4/3} k^{-5} \quad . \qquad (34)$$

The spectral laws (34) will fall by collisionless dissipation at a critical wave number

$$k_D = (a^2 \, \mathcal{E}_\lambda / \lambda^5)^{1/6} \quad , \qquad (35)$$

while the life time of the eddy of this critical size is

$$t = (J\omega_c)^{-1/3} \quad . \qquad (36)$$

14. DISCUSSIONS

14.1. Dimensional Considerations

The second of the spectral laws (34) for a collisionless plasma seems to agree quite well with experimental results. It would be important to further check experimentally the amplitude, the critical number of the fall (35), and the life time (36), especially as to their dependence on the rate of turbulent dissipation $\mathcal{E}_\lambda$.

As the density spectrum G assumes the dimension

$$G = \frac{1}{a^2} \frac{\ell^5}{t^4} ,$$

a spectral law

$$G = \frac{1}{a^2} \omega_c^4 k^{-5} \tag{37}$$

has been suggested for experimental usage[6,7], by choosing

$$t = \omega_c^{-1} ,$$

while the correct life time should be

$$t = (J\omega_c)^{-1/3} ,$$

yielding the spectral laws (34). The empirical law (37) appears to be not generally acceptable, as it is doubtful that the amplitude of the density fluctuations could be independent of any turbulent strength. Since ω_c^{-1} is not the only scale of time, the dimensional formulation of the spectral composition, without a dynamical foundation, suffers from the great arbitrariness in the multitude of choices of plausible formulas.

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[6] F.F. Chen, Phys. Rev. Letters 15, 381 (1965).

[7] N. D'Angelo and L. Enrigues, Phys. Fluids 9, 2290 (1966).

14.2. Density and Potential Fluctuations

In most experiments it has been assumed that the density fluctuations behave like the potential fluctuations. This is indeed true in the present system of cold ions in a bath of hot electrons where the wave lengths are greater then the Debye length. More specifically, if G assumes the k^{-5} spectral law, it will be so with the spectrum of density $\ln(N+n)$. However, in a one-fluid model, where the electrons are absent, the Poisson equation indicates that the term involving the Debye wave number drops, and we would have

$$k^4 G = \left(\frac{4\pi e^2}{Ma}\right)^2 G_n \quad ,$$

where G_n is the spectrum of $\langle n^2 \rangle$, giving the law

$$G_n = \text{const} \;\; k^{-1} \quad ,$$

corresponding to

$$G = \text{const} \;\; k^{-5} \quad .$$

14.3. Turblence in atmosphere and ocean

The system (2) is formally analogous to the Riemann[1] equations with the addition of rotation and dissipation, for a compressible gas with a ratio of specific heats equal to unity. The turbulent motion in atmosphere with temperature fluctuations and under a Coriolis force is governed by the same system, and shows under certain circumstances a spectral law (31a)

$$k^{-5}$$

in agreement with observation. The fluctuations in velocity and in height on the waters of the ocean with a Coriolis force also satisfy a similar system of equations. Those analogies suggest that the method of solving the nonlinear system of equations of turbulence and even some of the results presented here, may be applicable to such problems, and perhaps may pave the way of approaching the more general problem of compressible turbulence.

14.4. Collisionless dissipation mechanism

As shown by the Navier-Stokes equation of motion and the equation of continuity for a compressible fluid, the molecular viscosity provides a collisional dissipation for the turbulent motion, but since the equation of continuity does not contain a molecular diffusion, one asks what could be a dissipation mechanism for the density fluctuations. By means of the cascade decomposition, proposed in Sec. 4, an electrostatic diffusion by collective fluctuations can account for a collisionless dissipation. Like in the collisional case, the latter dissipation

$$\lambda J_0 \quad \text{or} \quad \lambda J$$

is also equal to the product of a vorticity by a diffusion coefficient. The diffusion coefficient λ is found to agree with the Bohm diffusion[5].

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